# The structure of conserved charges in open spin chains

M P Grabowski and P Mathieu

Département de Physique, Université Laval, Québec, Canada G1K 7P4

**Abstract.** We study the local conserved charges in integrable spin chains of the XYZ type with non-trivial boundary conditions. The general structure of these charges consists of a bulk part, whose density is identical to that of a periodic chain, and a boundary part. In contrast with the periodic case, only charges corresponding to interactions of even number of spins exist for the open chain. Hence, there are half as many charges in the open case as in the closed case. For the open spin-$\frac{1}{2}$ XY chain, we derive the explicit expressions of all the charges. For the open spin-$\frac{1}{2}$ XXX chain, several lowest-order charges are presented and a general method of obtaining the boundary terms is indicated. In contrast with the closed case, the XXX charges cannot be described in terms of a Catalan tree pattern.

## 1. Introduction

Recently we have obtained the structure of the local conserved charges for the XXX chain [1] and some of its generalizations: the isotropic $su(N)$ spin chain and the octonionic model [2]. Explicit expressions—in the form of a Catalan tree pattern—have been derived for all the charges. Different extensions of these results can be considered. In the present work we study the modification of the Catalan tree pattern for open (finite) chains†.

As for closed chains, there is a transfer matrix formalism for open chains from which the conserved quantities can be obtained by power expansion in terms of the spectral parameter. For closed chains, this approach is not an effective way of deriving the explicit expressions of the conserved charges. In the open case, the situation is even worse as the open transfer matrix is, roughly speaking, the trace of the square of the closed chain monodromy matrix (up to boundary terms). A somewhat surprising consequence of this squaring process is that half of the local conserved charges of the closed chain disappear when the chain is cut open. More precisely, as we will show below, the expression for the open conserved charges contain a bulk part and a boundary part. In the infinite chain limit, the boundary part is irrelevant: the bulk part is thus necessarily the full closed chain charge expression. Hence, a charge exists in the open case only if suitable boundary terms can be added to the bulk part to preserve the commutativity of the total charge with the Hamiltonian.

It turns out that conservation preserving boundary terms do not always exist. For instance, for the XY closed chain, there are two infinite sequences of local charges, that is, $\{H_n^{(1)}\}$ and $\{H_n^{(2)}\}$, for all integer values of $n$, where $n$ indicates the highest number of adjacent spin factors of the charge density. In the open case, there are no appropriate boundary counterterms for half of these charges. There remains only one charge for each value of $n$. For the closed XXX chain, we know that there exists one local charge $H_n$ for

† As far as the pattern of charges is concerned, the infinite open chain is not distinguishable from the closed case.

each positive integer $n$. However, for the open XXX case, only the charges with $n$ even survive. In particular, $H_3$ does not exist!

A major technical difficulty encountered in deriving explicit expressions for the conserved charges for open chains is caused by the non-existence of a boost operator, whose commutation with the Hamiltonian would generate the higher-order charges. As a result, there is no systematic and effective way of obtaining the expressions for the first few charges. In spite of this, one could try to find them by brute force and see whether an underlying pattern emerges. From the expressions of $H_4$ and $H_6$ obtained for the XXX open chain, this would seem to be so: their boundary parts can be written in terms of the same spin polynomials as their bulk part and they can actually be expressed in terms of lower-order 'bulk charges' localized at the boundaries. However, this is not a generic feature: $H_8$ cannot be written in this polynomial basis.

The paper is organized as follows. In the next section, we review the transfer matrix formalism. This is used to show the absence of odd degree local conserved charges. A second argument is worked out in the following section, from a computational approach: the charge is written as a sum of a (known) bulk piece and a set of boundary counterterms to be fixed. We show that in $H_n$ the counterterms must cancel a Catalan-tree bulk-type charge localized at each boundaries. We then demonstrate that no such terms can be found for $H_3$ and then for all $H_{2n+1}$. The explicit form of the first few non-trivial charges is presented. Unfortunately the simple polynomial basis used to describe the closed XXX charge is inadequate in the open case, thus making it very hard to find a general expression for all the charges. Straightforward generalizations and conclusions are reported in the final section.

## 2. The general structure of local conservation laws in open chains

In this section we briefly recall the transfer matrix formalism and use it to determine the general structure of conservation laws in open spin chains. This will be done first for the relatively simple spin-$\frac{1}{2}$ XXX model, and later the extension of the argument to more general integrable chains will be indicated.

The integrable spin chain with non-trivial boundary conditions [3] is described by the transfer matrix

$$\mathbf{t}_o(u) = \mathrm{tr}(\mathbf{K}_+(u)\mathbf{T}(u)\mathbf{K}_-(u)\mathbf{T}^{-1}(-u)). \tag{2.1}$$

$u$ is the spectral parameter and $\mathbf{T}(u)$ is the monodromy matrix, whose trace gives the usual (closed) chain transfer matrix:

$$\mathbf{t}_c(u) = \mathrm{tr}\,\mathbf{T}(u) \tag{2.2}$$

(the subscripts 'o' and 'c' refer to open and closed cases, respectively). The monodromy matrix is constructed from the basic $R$-matrix of the model

$$\mathbf{T}(u) = R_{N0} \cdots R_{10}. \tag{2.3}$$

The index $i = 1, \ldots, N$ labels a vector space at site number $i$ and 0 refers to internal or auxiliary space, over which the trace in (2.1) is taken. $\mathbf{K}_+$ and $\mathbf{K}_-$ are left and right boundary matrices, respectively. For the XYZ model, the most general form of these matrices compatible with integrability has been derived in [4]. For free open boundaries, to which we confine ourselves for the most part of this work, $\mathbf{K}_\pm = \mathbf{I}$, the $2 \times 2$ identity matrix.

The transfer matrix can be used as a generating function for the conserved quantities. It follows from the Yang–Baxter equation that any derivative of the transfer matrix commutes

with the Hamiltonian (which is, by construction, the first logarithmic derivative of the transfer matrix, taken at the particular value $u = 0$ of the spectral parameter). The quantities obtained in this way are, however, nonlocal, i.e. they contain spin interactions at distances growing arbitrarily when the length of the chain is increased. To obtain local conserved quantities out of the transfer matrix, one needs to take logarithmic derivatives [5]. The argument of Lüscher, showing the local nature of the logarithmic derivatives of $\mathbf{t}_c(u)$ at $u = 0$, remains valid for open chains. However, there are two new features in the open case as compared to the periodic one. First, translational invariance is obviously lost. Second, and more surprisingly, all charges with an odd number of interacting spins disappear!

We will first concentrate on the spin-$\frac{1}{2}$ XXX chain, whose $R$-matrix takes the simple form

$$R = u\mathbf{I} + P \tag{2.4}$$

where $P$ is the permutation operator. The derivatives of the transfer matrix of the closed chain are

$$\mathbf{t}_c^{(n)}(0) \equiv \left.\frac{d^n \mathbf{t}_c(u)}{du^n}\right|_{u=0} = n! \sum_{|\mathcal{C}|=N-n} \mathcal{P}_R(\mathcal{C}) \tag{2.5}$$

where the sum goes over all ordered clusters $\mathcal{C}$ of $(N-n)$ points, that is, sets $\{i_1, \ldots, i_{N-n}\}$ with $i_1 < i_2 < \cdots < i_{N-n}$. $\mathcal{P}_R(\mathcal{C})$ and $\mathcal{P}_L(\mathcal{C})$ (appearing below) denote the cyclic permutation of the spins of the sequence $\mathcal{C}$ to the right or to the left, respectively. In particular,

$$\mathbf{t}_c(0) = \mathcal{P}_R(\Lambda) \tag{2.6}$$

where $\Lambda = \{1, 2, \ldots, N\}$. The logarithmic derivatives of the transfer matrix yield local charges. In particular, using

$$\mathbf{t}_c^{-1}(0) = \mathcal{P}_L(\Lambda) \tag{2.7}$$

it can be easily seen that

$$\mathbf{t}_c^{-1}(0)\mathbf{t}_c^{(1)}(0) = \sum_{i=1}^{N} \mathcal{P}_R(\{i, i+1\}) = \sum_{i=1}^{N} P_{i,i+1}. \tag{2.8}$$

The permutation operator above can be written as

$$P_{i,i+1} = \tfrac{1}{2}(1 + \sigma_i \cdot \sigma_{i+1}) \tag{2.9}$$

(where $\sigma_i$ denotes a Pauli matrix at site $i$) and thus (2.8), up to a constant, coincides with the Hamiltonian of the periodic XXX chain.

Let $Q_n^c$ denote the $(n-1)$th logarithmic derivative of $\mathbf{t}_c(u)$ at $u = 0$:

$$Q_n^c = \frac{d^{n-1}}{du^{n-1}} \ln \mathbf{t}_c(u)|_{u=0}. \tag{2.10}$$

In terms of $v_k$ defined as

$$v_k = \mathbf{T}^{-1}(0)\mathbf{T}^{(k)}(0) \tag{2.11}$$

we have

$$Q_n^c = \operatorname{tr} v_{n-1} + \operatorname{tr} p_{n-1}(v_1, \ldots, v_{n-2}) \tag{2.12}$$

where $p_{n-1}$ is a homogeneous polynomial (of degree $n-1$) in $v_1, \ldots, v_{n-2}$. These charges are translationally invariant and can be written in the form

$$Q_n^c = \sum_{j=1}^{N} q_n(j) \tag{2.13}$$

where the density $q_n(j)$ contains interactions of at most $n$ spins on sites $\{j, j+1, \ldots, j+n-1\}$ (addition understood modulo $N$).

Consider now the chain with open free boundaries ($\mathbf{K}_\pm = \mathbf{I}$). Conserved charges may be obtained from the expansion of the logarithm of the transfer matrix $\mathbf{t}_o(u)$:

$$Q_n^o = \frac{d^{n-1}}{du^{n-1}} \ln \mathbf{t}_o(u)|_{u=0}. \tag{2.14}$$

We will now argue that for $n$ even, the leading term—that with the greatest number of interacting spins—in the above expression is the same as in the closed case. Taking the trace over the two-dimensional auxiliary space, we obtain

$$\mathbf{t}_o(0) = 2\mathbf{I} \tag{2.15}$$

where $\mathbf{I}$ is the $2^N \times 2^N$ identity matrix. Therefore, $\mathbf{t}_o^{-1}(0)$ is simply $\frac{1}{2}$, and

$$Q_n^o = \tfrac{1}{2}\mathbf{t}_o^{(n-1)}(0) + p_{n-1}(\tfrac{1}{2}\mathbf{t}_o^{(1)}(0), \ldots, \tfrac{1}{2}\mathbf{t}_o^{(n-1)}(0)) \tag{2.16}$$

where $p_{n-1}$ is the same polynomial as in (2.12). Consider first $n$ to be even. From (2.1) we get

$$\mathbf{t}_o^{(n-1)}(0) = 2 \operatorname{tr} v_{n-1} + \operatorname{tr} r_{n-1} \tag{2.17}$$

where $r_{n-1}$ is a homogeneous polynomial of degree $n-1$ in the variables $v_1, \ldots, v_{n-2}$. Note that (2.16) and (2.12) both contain the same local term

$$\operatorname{tr} v_{n-1} \sim \sum_i \mathcal{P}_R(\{i, i+1, \ldots, i+n-2\}) \tag{2.18}$$

(with the addition understood modulo $N$ for the closed chain). For the closed XXX chain, the set of local conserved charges obtained from the transfer matrix has been shown to be complete (see [6]). This means that any local conserved quantity in the closed chain must be a linear combination of the logarithmic derivatives of the transfer matrix $\mathbf{t}_c(u)$. In particular, this implies that given the leading term in $Q_n^c$ (the one describing $n$-spin interactions), there is a unique set of subleading terms ensuring the commutativity of $Q_n^c$ with the Hamiltonian. The introduction of non-trivial boundary conditions can be viewed as a perturbation of the original periodic chain† and this can only reduce the number of conservation laws. The key point is that, as can be seen from (2.16) and (2.12) for $n$ even, the leading term of a charge $Q_n^o$ must be the same as for the closed chain. Consider now the infinite limit of the open chain. Obviously, in this limit (2.10) must coincide with some function of conserved quantities for the infinite periodic chain, but the only local conserved quantities in this case are linear combinations of the charges $Q_n^c$. Therefore, the bulk density of the open chain $n$-spin charge $Q_n^o$ must be the same as the density of $Q_n^c$. It follows that for $n = 2m$, $Q_{2m}^o$ has the form

$$Q_{2m}^o = Q_{2m}^{\text{bulk}} + q_{2m}^L + q_{2m}^R \tag{2.19}$$

where

$$Q_{2m}^{\text{bulk}} = \sum_{j=1}^{N-2m+1} q_{2m}(j) \tag{2.20}$$

(i.e. we take only clusters that do not 'jump' over the boundary) and $q_{2m}^L$ ($q_{2m}^R$) stands for boundary terms at the left (right) boundary. As we will show in section 3, the boundary terms involve at most the $2m - 2$ spins adjoining the boundary.

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† If $H_c$ denotes the Hamiltonian of a periodic spin chain with nearest-neighbour interactions, $H_c = \sum_{j=1}^{N} h_{j,j+1}$, by adding the single perturbation term $-h_{1,N}$ one obtains the open-ended chain $H_o = H_c - h_{1,N}$.

In contrast, when $n$ is odd, i.e. $n = 2m + 1$, there is no totally local term in $Q_n^o$:

$$Q_{2m+1}^o = p_{2m}(\tfrac{1}{2}\mathbf{t}_o^{(1)}, \ldots, \tfrac{1}{2}\mathbf{t}_o^{(n-1)}) \tag{2.21}$$

and the $n$-spin contribution vanishes. In consequence, the bulk part of $Q_{2m+1}^o$ is zero, implying that for $n$ odd, $Q_n^o$ must be trivial.

The absence of odd-order charges is in fact a simple consequence of the chain reversal symmetry of the open chain, i.e. its invariance under the transformation

$$\sigma_n \to \sigma_{N-n+1}. \tag{2.22}$$

This symmetry is specific to the open case: the transfer matrix of the periodic XXX chain is not invariant under (2.22):

$$\text{tr}(R_{N,0}(u) R_{N-1,0}(u) \ldots R_{1,0}(u)) \neq \text{tr}(R_{1,0}(u) R_{2,0}(u) \ldots R_{N,0}(u)). \tag{2.23}$$

Indeed, it is easy to see from the explicit form of the local conserved quantities of the XXX chain (see [1]), that the logarithmic derivatives of $\mathbf{t}_c(u)$ of odd degrees (corresponding to charges with an even number of spins) do not change under (2.22), while those of even degree (corresponding to odd number of spins) change sign.

In contrast, the transfer matrix of the open XXX chain is invariant under the operation (2.22). Using

$$R_{n,0}^{-1}(-u) = -\frac{1}{1 + u^2} R_{n,0} \tag{2.24}$$

$\mathbf{t}_o(u)$ can be written in the form

$$\mathbf{t}_o(u) = \text{tr}(R_{N,0} \ldots R_{1,0}(u) R_{1,0}^{-1}(-u) \ldots R_{N,0}^{-1}(-u)$$

$$= \frac{(-1)^N}{(1 + u^2)^N} \text{tr}(R_{N,0}(u) \ldots R_{2,0}(u) R_{1,0}^2(u) R_{2,0}(u) \ldots R_{N,0}(u)). \tag{2.25}$$

Under (2.22), it transforms as

$$\mathbf{t}_o(u) \to \frac{(-1)^N}{(1 + u^2)^N} \text{tr}(R_{1,0}(u) \ldots R_{N-1,0}(u) R_{N,0}^2(u) R_{N-1,0}(u) \ldots R_{1,0}(u)) \tag{2.26}$$

which, due to the cyclic property of the trace, is equal to (2.25). This symmetry excludes thus the possibility of building-up the bulk density of $Q_{2m+1}^o$ from the densities of odd-spin charges $Q_{2m+1}^c$ since their sign is changed under (2.22). Hence, there are no odd-spin charges for the open XXX chain.

The above argument for the spin-$\frac{1}{2}$ XXX chain may be similarly applied to the open XYZ chain. As before, the key step is to observe that, for $n$ even, (2.16) and (2.12) both contain the same local leading term with $n$ spins interacting. In the XXX case, one could use the completeness property of the system of charges generated by the transfer matrix of the periodic chain to show that the bulk density in the open case must coincide with the density of the periodic chain. To our knowledge, the completeness of the system of charges (2.10) for the general XYZ case has not been established. However, considering the infinite limit of the open chain, one may conclude that the bulk density of the conserved charge (2.10) must coincide with the density of the corresponding closed chain expression, modulo the densities of additional conserved charges (which do not have to be generated by the transfer matrix (2.2)). Therefore, the set of conserved quantities generated from the transfer matrix still has the general structure (2.19), but the bulk density may now, *a priori*, contain additional contributions. For $n$ odd, there is no $n$-spin local term in (2.16), and in consequence the bulk density of $Q_{2m+1}$ must vanish. As for the XXX case, this can

be viewed as a consequence of the chain reversal symmetry (2.22). Recall that the basic $R$-matrix of the XYZ model satisfies the unitarity requirement

$$\mathbf{R}(u)\mathbf{R}(-u) = \rho(u)\mathbf{I} \tag{2.27}$$

where $\rho(u)$ is some scalar function. Using (2.27), the transfer matrix of the open XYZ chain can be rewritten in a form which is manifestly invariant under (2.22):

$$\mathbf{t}_{\mathrm{o}}(u) = \mathrm{tr}(R_{N,0}(u)\ldots R_{2,0}(u)R_{1,0}(u)R_{1,0}^{-1}(-u)R_{2,0}^{-1}(-u)\ldots R_{N,0}^{-1}(-u))$$

$$= \frac{1}{\rho(u)^N}\,\mathrm{tr}(R_{1,0}(u)\ldots R_{N-1,0}(u)R_{N,0}^2(u)R_{N-1,0}(u)\ldots R_{1,0}(u)). \tag{2.28}$$

Note that the above argument does not exclude a possible existence of additional odd-spin conserved charges not given by the formula (2.10). Such a situation indeed exists for the XY model (see section 4). However, this is not expected in the non-degenerate XYZ case, for which the system of charges generated from the transfer matrix is likely to be complete.

## 3. The calculation of the conserved charges for the open XXX model

The model-defining Hamiltonian for the XXX chain with free open boundaries is†

$$H_2 = \sum_{i=1}^{N} \sigma_i \cdot \sigma_{i+1} \tag{3.1}$$

where $\sigma$ stands for the vector $(\sigma^x, \sigma^y, \sigma^z)$, $\sigma$'s being the usual Pauli matrices. As we have seen in the previous section, higher-order charges will have a bulk part and a boundary part. It is convenient to redefine the charges so that in the bulk they will be described by the Catalan tree pattern [1]. (This corresponds to a change of basis from $\{Q_n\}$ to $\{H_n\}$ where such that $H_n$ does not contain any linear combination of lower order $H_m$'s. This change of basis corresponds to taking linear combinations of the logarithmic derivatives of the transfer matrix.) We thus look for charges in the form

$$H_n = H_n^{\mathrm{bulk}} + h_n^{\mathrm{L}} + h_n^{\mathrm{R}} \tag{3.2}$$

where $h_n^{\mathrm{L}}$ and $h_n^{\mathrm{R}}$ are boundary terms located at the left and right extremity of the chains, respectively (their precise extension will be evaluated later); these are the unknowns to be determined. With the above ansatz, we assume the chain to be sufficiently long to prevent a mixing of boundary terms from opposite boundaries. Recall that the local integrals of motion for the closed XXX model can be expressed as linear combinations of the $F_{n,k}^{\mathrm{c}}$'s defined as

$$F_{n,k}^{\mathrm{c}} = \sum_{\mathcal{C}\in\mathcal{C}^{(n,k)}} f_n(\mathcal{C}). \tag{3.3}$$

$\mathcal{C}^{(n,k)}$ stands for the set of all $n$ ordered sites with $k$ holes (i.e. the sites are not necessarily adjacent) and $\mathcal{C}$ is some cluster among this set. $f_n(i_1, i_2, \ldots, i_n)$ is defined as

$$\begin{aligned}
f_0 &= f_1 = 0 \\
f_2 &= \sigma_{i_1} \cdot \sigma_{i_2} \\
f_3 &= (\sigma_{i_1} \times \sigma_{i_2}) \cdot \sigma_{i_3} \\
&\cdots \\
f_n &= (\cdots((\sigma_{i_1} \times \sigma_{i_2}) \times \cdots) \cdot \sigma_{i_n}.
\end{aligned} \tag{3.4}$$

† From now on we omit the superscript 'o' from conserved charges.

(Notice that the parentheses are nested toward the left.) The closed chain conservation laws $H_n^c$'s are

$$H_n^c = F_{n,0}^c + \sum_{k=1}^{[n/2]-1} \sum_{\ell=1}^{k} \alpha_{k,\ell} F_{n-2k,\ell}^c \tag{3.5}$$

where the coefficients $\alpha_{k,\ell}$ are the generalized Catalan numbers

$$\alpha_{k,\ell} = \binom{2k-\ell}{k-\ell+1} - \binom{2k-\ell}{k-\ell-1}. \tag{3.6}$$

The bulk part of an open-chain conserved charge (of even order) is then given simply by restricting the corresponding closed chain expression to those clusters which do not 'jump over the border'. More precisely, let $\mathcal{O}^{(n,k)}$ denote the subset of $\mathcal{C}^{(n,k)}$ obtained by removing the clusters which overlap the borders. (For instance, for $N = 6$, $\mathcal{O}^{(3,3)} = (1, 2, 6), (1, 3, 6), (1, 4, 6), (1, 5, 6)$.) Denoting

$$F_{n,k}^o = \sum_{\mathcal{C} \in \mathcal{O}^{(n,k)}} f_n(\mathcal{C}) \tag{3.7}$$

$H_n^{\text{bulk}}$ is given by formula (3.5) with $F_{n,k}^c$ replaced by $F_{n,k}^o$.

For the purpose of determining the boundary terms, we can focus on a single extremity, or equivalently, consider a semi-infinite chain ($N \to \infty$, so that $i = 1, 2, \ldots$). The term $h_n^L$ can then be characterized as follows. It is designed to cancel, in the commutator $[H_2, H_n]$, those terms that would be cancelled for an infinite chain ($i \in \mathbb{Z}$) by the contribution of the 'link' $\sigma_0 \cdot \sigma_1$. In the infinite chain commutator $[\sigma_0 \cdot \sigma_1, H_n^{(\infty)}]$, these are precisely those resulting terms that live in the semi-infinite chain, i.e. on sites $i \geqslant 1$. Our first step is thus to evaluate this commutator:

$$[H_2, H_n^{\text{bulk}}] = -[\sigma_0 \cdot \sigma_1, H_n^{(\infty)}]|_{i \geqslant 1} \equiv \mathcal{R}_n^L \tag{3.8}$$

where $\mathcal{R}_n^L$ stands for left remainder. This expression is readily evaluated: it is a sort of anti-boost, that is, the inverse of the boost action that generates $H_{n+1}$ from $H_n$. In the boost operation, $[H_2, H_n]$ gives, in addition to $H_{n+1}$, some lower order local charges. This turns out not to be the case for the anti-boost calculation. We find the remarkably simple result

$$[\sigma_0 \cdot \sigma_1, H_n^{(\infty)}]|_{i \geqslant 1} = 4i H_{n-1}^{\text{bulk}}|_{\text{L-bdry}} \tag{3.9}$$

where $H_{n-1}^{\text{bulk}}$ restricted to the left boundary contains only those terms of $H_{n-1}^{\text{bulk}}$ whose leftmost spin is at site 1. This is proved below.

Let us first introduce a schematic way of describing the related calculations. To represent the monomial $f_n(i_1, i_2, \ldots, i_n)$ we use a sequence of dots, with black dots corresponding to occupied sites. For our purposes, we will only need sequences of dots based at site 0 (that is, the first dot represents $i_1 = 0$). For instance $f_6(0, \ldots, 5)$ and $f_4(0, 1, 3, 5)$ are represented, respectively, by

$$f_6(0, \ldots, 5) = ((((\sigma_0 \times \sigma_1) \times \sigma_2) \times \sigma_3) \times \sigma_4) \cdot \sigma_5 \sim \bullet\bullet\bullet\bullet\bullet\bullet \tag{3.10}$$

and

$$f_4(0, 1, 3, 5) = ((\sigma_0 \times \sigma_1) \times \sigma_3) \cdot \sigma_5 \sim \bullet\bullet\circ\bullet\circ\bullet. \tag{3.11}$$

The commutation relation

$$[\sigma_0 \cdot \sigma_1, (\sigma_0 \times \sigma_1) \cdot \sigma_2] = 4i(\sigma_1 \cdot \sigma_2 - \sigma_0 \cdot \sigma_2) \tag{3.12}$$

will then be represented by the following diagram:

$$\overset{\sqcap}{\bullet\bullet}\ \bullet = 4i\ \circ\bullet\bullet\ -\ 4i\ \bullet\circ\bullet. \tag{3.13}$$

The contraction in this diagram indicates taking the value of the commutator of the 'link' (i.e. the $\sigma_0 \cdot \sigma_1$ operator) with the $f$ polynomial represented by a sequence of dots. In the following, we will implement the projection onto the semi-infinite chain by setting to zero all spin factors at site $i \leqslant 0$, that is, by dropping the terms whose first dot is occupied. This projection will be indicated by an arrow. For example, in (3.13) it amounts to setting to zero the term $\sigma_0 \cdot \sigma_2$:

$$\overset{\sqcap}{\bullet} \; \bullet \; \bullet \;\; = 4i \; \circ \bullet \bullet \, . \tag{3.14}$$

Here are sample calculations of $[\sigma_0 \cdot \sigma_1, H_n^{(\infty)}]|_{i \geqslant 1}$. For $n = 4$,

$$H_4^{(\infty)} = F_{4,0} + F_{2,1}. \tag{3.15}$$

It is clear that the only terms in $H_4$ that will contribute to this commutator, after the projection, are those whose leftmost spin is at site 0. The contribution of $F_{4,0}$ is

$$\overset{\sqcap}{\bullet} \; \bullet \; \bullet \bullet \;\; \to 4i \; \circ \bullet \bullet \bullet \, . \tag{3.16}$$

The commutator with $F_{2,1}$ does not survive the projection:

$$\overset{\sqcap}{\bullet} \; \circ \; \bullet \;\; = 2i \; \bullet \circ \bullet \;\; \to 0. \tag{3.17}$$

The result is exactly the part of $F_{3,0}$ based at the left boundary, that is,

$$\mathcal{R}_4^{\mathrm{L}} = -4i \, H_3^{\mathrm{bulk}}|_{\text{L-bdry}}. \tag{3.18}$$

For $n = 5$,

$$H_5^{(\infty)} = F_{5,0} + F_{3,1} \tag{3.19}$$

we have

$$\overset{\sqcap}{\bullet} \; \bullet \; \bullet \bullet \bullet \;\; \to 4i \; \circ \bullet \bullet \bullet \bullet \tag{3.20}$$

and

$$\overset{\sqcap}{\bullet} \; \bullet \; \circ \bullet \;\; \to 4i \; \circ \bullet \circ \bullet \, . \tag{3.21}$$

We thus obtain

$$\mathcal{R}_5^{\mathrm{L}} = -4i \, H_4^{\mathrm{bulk}}|_{\text{L-bdry}}. \tag{3.22}$$

Having worked out these examples, let us return to the proof of the general identity (3.9). The contribution of a term $F_{n,k}$ to the commutator is clearly $4i \, F_{n-1,k}$ restricted to the left boundary: indeed, only monomials of the form $f_n(0, 1, \ldots, n-1)$ contribute in the commutator and after commutation and projection, each such term is transformed into $4i f_{n-1}(1, \ldots, n-1)$, which builds up $F_{n-1,k}$ restricted to clusters starting at site 1. (The number of holes is unaffected in this process.) If $n$ is odd, the structure of the Catalan tree is not modified and $H_n^{(\infty)}$ is mapped to $4i \, H_{n-1}^{\mathrm{bulk}}|_{\text{L-bdry}}$. If $n$ is even, the terms $F_{2,k}$ do not contribute and we simply strip off the last row of the Catalan tree of $H_n^{(\infty)}$ to get the tree of $H_{n-1}^{(\infty)}$, restricted to the left boundary.

Up to this point, we have only found what the boundary terms $h_n^{\mathrm{L}}$ have to cancel when commuted with $H_2$: we require

$$[H_2, h_n^{\mathrm{L}}] = -\mathcal{R}_n^{\mathrm{L}} = 4i \, H_{n-1}^{\mathrm{bulk}}|_{\text{L-bdry}}. \tag{3.23}$$

The existence of a conserved charge is thus reduced to the existence a suitable $h_n^{\mathrm{L}}$ satisfying this equation.

In this framework, the non-existence of $H_3$ is simply established: there are no $h_3^L$, build up with 1 or 2 spin factors, that could cancel $4i\sigma_1 \cdot \sigma_2$ upon commutation with $\sigma_1 \cdot \sigma_2 + \sigma_2 \cdot \sigma_3$. The same argument also implies the non-existence of any odd charge, simply because $H_{2n+1}^{(\infty)}$ contains a three-spin piece and $h_{2n+1}^L$ would have to contain exactly the same counterterm required to make a charge $H_3$.

Equation (3.23) implies that the boundary part in $H_n$ may contain only interactions of up to $n-2$ spins adjoining the boundary. For the spin-$\frac{1}{2}$ chain, the most general term containing multi-spin interactions is a multilinear polynomial in spin variables, which can be equivalently represented in the permutation basis as a linear combination of permutations of the $n-1$ spins at the boundary (with $(n-1)!$ arbitrary coefficients). These coefficients are then fixed by enforcing the condition (3.23). The first two boundary terms are:

$$h_4^L = 4\mathcal{P}_L(1, 2) \tag{3.24}$$
$$h_6^L = 4(-\mathcal{P}_L(1, 2, 3, 4) - \mathcal{P}_L(4, 3, 2, 1) + \mathcal{P}_L(1, 2, 4, 3) + \mathcal{P}_L(2, 1, 3, 4))$$
$$+ 8\mathcal{P}_R(1, 3) - 8\mathcal{P}_R(2, 3). \tag{3.25}$$

A particularly useful set of polynomials in spin variables is provided by the $f_n(\mathcal{C})$'s (where $\mathcal{C}$ is an ordered cluster ($i_1 < \cdots < i_n$)), introduced above as the building blocks of the bulk part of $H_n$. They can be expressed in terms of permutations as follows:

$$f_n(\{i_1, i_2, \ldots, i_n\}) = 2(-i)^{n-2}[\ldots[[P_{i_n i_{n-1}}, P_{i_{n-1} i_{n-2}}], P_{i_{n-2} i_{n-3}}]\ldots, P_{i_2 i_1}] \tag{3.26}$$

with

$$f_2(i_1, i_2) = 2P_{i_1 i_2} - 1. \tag{3.27}$$

In particular,

$$f_3(i_1, i_2, i_3) = 2i(\mathcal{P}_L(i_1, i_2, i_3) - \mathcal{P}_L(i_3, i_2, i_1))$$
$$f_4(i_1, i_2, i_3, i_4) = -2(\mathcal{P}_L(i_1, i_2, i_3, i_4) + \mathcal{P}_L(i_4, i_3, i_2, i_1)$$
$$- \mathcal{P}_L(i_1, i_2, i_4, i_3) - \mathcal{P}_L(i_3, i_4, i_2, i_1))$$
$$f_6(i_1, i_2, i_3, i_4, i_5, i_6) = 2(\mathcal{P}_L(i_1, i_2, i_3, i_4, i_5, i_6) - \mathcal{P}_L(i_1, i_2, i_3, i_4, i_6, i_5)$$
$$- \mathcal{P}_L(i_1, i_2, i_3, i_5, i_6, i_4) + \mathcal{P}_L(i_1, i_2, i_3, i_6, i_5, i_4)$$
$$- \mathcal{P}_L(i_1, i_2, i_4, i_5, i_6, i_3) + \mathcal{P}_L(i_1, i_2, i_4, i_6, i_5, i_3) \tag{3.28}$$
$$+ \mathcal{P}_L(i_1, i_2, i_5, i_6, i_4, i_3) - \mathcal{P}_L(i_1, i_2, i_6, i_5, i_4, i_3)$$
$$- \mathcal{P}_L(i_1, i_3, i_4, i_5, i_6, i_2) + \mathcal{P}_L(i_1, i_3, i_4, i_6, i_5, i_2)$$
$$+ \mathcal{P}_L(i_1, i_3, i_5, i_6, i_4, i_2) - \mathcal{P}_L(i_1, i_3, i_6, i_5, i_4, i_2)$$
$$+ \mathcal{P}_L(i_1, i_4, i_5, i_6, i_3, i_2) - \mathcal{P}_L(i_1, i_4, i_6, i_5, i_3, i_2)$$
$$- \mathcal{P}_L(i_1, i_5, i_6, i_4, i_3, i_2) + \mathcal{P}_L(i_1, i_6, i_5, i_4, i_3, i_2)).$$

The boundary terms $h_4^L$ and $h_6^L$ have a simple form when expressed in the $f_n(\mathcal{C})$ basis. Define

$$f_{n,k}(j) = \sum_{\mathcal{C} \in \mathcal{C}^{(n,k)}(j)} f_n(\mathcal{C}) \tag{3.29}$$

where $\mathcal{C}^{(n,k)}(j)$ stands for the set of $(n, k)$ clusters based at site $j$ (recall that $k$ indicates the number of holes). Then we have

$$h_4^L = 2f_{2,0}(1) \tag{3.30}$$
$$h_6^L = 2f_{4,0}(1) + 4f_{2,1}(1) - 4f_{2,0}(2). \tag{3.31}$$

One might hope that all the higher boundary terms could be similarly expressed in terms of the polynomials $f_{n,k}(j)$, which would significantly reduce the number of coefficients in the ansatz for the boundary part of $H_n$. It is easily seen to hold for the leading boundary term:

$$h_{2m}^{\mathrm{L}} = 2f_{2m-2,0}(1) + \text{lower orders.} \tag{3.32}$$

Unfortunately, this is not true in general for the subleading terms as can be seen from the exact expression for $h_8^{\mathrm{L}}$

$$h_8^{\mathrm{L}} = 2f_{6,0}(1) - 4f_{4,0}(2) + 2f_{4,1}(1) + 8f_{2,0}(3) + 2f_{2,0}(2)$$
$$+8f_{2,1}(1) + 2f_{2,2}(1) - 8f_{2,1}(2) + 2f_4(1, 3, 4, 5) - 2f_4(1, 3, 2, 4). \tag{3.33}$$

Since the last term corresponds to a *disordered* cluster, $h_8^{\mathrm{L}}$ cannot be rewritten as a linear combination of the simple polynomials $f_n(\mathcal{C})$ (with $\mathcal{C}$ ordered). Higher-order boundary terms are rather complicated and no simple pattern seems to emerge.

## 4. The XY chain and related models

In this section, we present the results for the conservation laws of the open chain of the XY type. The basic open XY model is defined by the Hamiltonian

$$H_{\mathrm{XY}} = \sum_{j=1}^{N-1} \lambda_x \sigma_j^x \sigma_{j+1}^x + \lambda_y \sigma_j^y \sigma_{j+1}^y. \tag{4.1}$$

As is well known, by the Jordan–Wigner transformation, the XY chain (regardless of the boundary conditions) can be reduced to a free-fermionic theory. Not surprisingly, the conserved charges for free-fermion chains have a particularly simple form. For periodic boundary conditions, the conservation laws have been described in [7–10] (see also [1]).

An interesting feature of the closed XY case is the existence of two independent families of conservation laws, which both persist when the model is perturbed by a perpendicular (i.e. in the $z$-direction) magnetic field. As we will see shortly, when the closed chain is cut open, only half of each family survives.

We consider first the particularly simple special case $\lambda_x = \lambda_y = 1$ (the XX model). We introduce the notation (for $n \geqslant 2$)

$$e_n^{\alpha\beta}(j) = \sigma_j^\alpha \sigma_{j+1}^z \cdots \sigma_{j+n-2}^z \sigma_{j+n-1}^\beta \tag{4.2}$$

and

$$\begin{aligned}
e_n^+(j) &= e_n^{xx}(j) + e_n^{yy}(j) & n \text{ even} \\
&= e_n^{xy}(j) - e_n^{yx}(j) & n \text{ odd}
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
e_n^-(j) &= e_n^{xy}(j) - e_n^{yx}(j) & n \text{ even} \\
&= e_n^{xx}(j) + e_n^{yy}(j) & n \text{ odd}
\end{aligned} \tag{4.4}$$

with

$$e_1^-(j) = -\sigma_j^z. \tag{4.5}$$

For the periodic XX chain, the two mutually commuting families of conserved charges are

$$^c H_n^\pm = \sum_{j=1}^N e_n^\pm(j). \tag{4.6}$$

In particular, there are two 'two-spin Hamiltonians':

$$^c H_2^+ = \sum_{j \in \Lambda} \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y \tag{4.7}$$

$$^{c}H_2^- = \sum_{j \in \Lambda} \sigma_j^x \sigma_{j+1}^y - \sigma_j^y \sigma_{j+1}^x. \tag{4.8}$$

$^{c}H_2^+$ is invariant under global parity transformations, i.e. $\sigma_j^a \to -\sigma_j^a$, while $^{c}H_n^-$ behaves like a pseudoscalar†.

The open chain versions of these Hamiltonians

$$H_2^\pm = \sum_{j=1}^{N-1} e_2^{(\pm)}(j) \tag{4.9}$$

no longer commute with each other:

$$[H_2^+, H_2^-] = 4i\sigma_N^z - 4i\sigma_1^z. \tag{4.10}$$

To each Hamiltonian, there corresponds a different infinite family of conserved charges. Their bulk parts are given by

$$H_n^{\pm;\text{bulk}} = \sum_{j=1}^{N-n+1} e_n^\pm(j). \tag{4.11}$$

The family of conservation laws for the scalar Hamiltonian is

$$\begin{aligned} H_n^{(1)} &= H_n^{+;\text{bulk}} + h_n^{+;\text{L}} + h_n^{+;\text{R}} && n \text{ even} \\ & \quad H_n^{-;\text{bulk}} + h_n^{-;\text{L}} + h_n^{-;\text{R}} && n \text{ odd} \end{aligned} \tag{4.12}$$

where

$$h_n^{\pm;\text{L}} = - \sum_{k=1}^{[n/2]-1} e_{n-2k}^\pm(k) \tag{4.13}$$

$$h_n^{\pm;\text{R}} = - \sum_{k=1}^{[n/2]-1} e_{n-2k}^\pm(N - k + 1). \tag{4.14}$$

For example,

$$h_3^{-;\text{L}} = -e_1^-(1) = \sigma_1^z$$
$$\begin{aligned} h_8^{+;\text{L}} &= -(e_6^+(1) + e_4^+(2) + e_2^+(3)) = -(\sigma_1^x \sigma_2^z \sigma_3^z \sigma_4^z \sigma_5^z \sigma_6^x + \sigma_1^y \sigma_2^z \sigma_3^z \sigma_4^z \sigma_5^z \sigma_6^y \\ &\quad + \sigma_2^x \sigma_3^z \sigma_4^z \sigma_5^x + \sigma_2^y \sigma_3^z \sigma_4^z \sigma_5^y + \sigma_3^x \sigma_4^x + \sigma_3^y \sigma_4^y). \end{aligned} \tag{4.15}$$

Obviously, $h_n^{\pm;\text{R}}$ can be obtained from $h_n^{\pm;\text{L}}$ via the chain reversal (2.22). The conservation of the family $H_n^{(1)}$

$$[H_2^+, H_n^{(1)}] = 0 \tag{4.16}$$

can be verified by a straightforward calculation, using

$$[H_2^+, e_n^\pm(j)] = \pm 2i(-1)^n(e_{n+1}^\pm(j-1) - e_{n+1}^\pm(j) + e_{n-1}^\pm(j+1) - e_{n-1}^\pm(j)) \tag{4.17}$$

where $1 < j < N$.

Similarly, one obtains the family of conservation laws for the open chain pseudoscalar Hamiltonian

$$\begin{aligned} H_n^{(2)} &= H_n^{-;\text{bulk}} + g_n^{-;\text{L}} + g_n^{-;\text{R}} && n \text{ even} \\ &= H_n^{+;\text{bulk}} + g_n^{+;\text{L}} + g_n^{+;\text{R}} && n \text{ odd} \end{aligned} \tag{4.18}$$

† $^{c}H_2^-$ is a special case of the Dzyaloshinski–Moriya interaction [11].

where

$$g_n^{\pm;L} = \sum_{k=1}^{[n/2]-1} (-1)^{k+1} e_{n-2k}^{\pm}(k) \tag{4.19}$$

and $g_n^{\pm;R}$ is obtained from $g_n^{\pm;L}$ by chain reversal. For example,

$$g_8^{-;L} = e_6^-(1) + e_4^-(2) + e_2^-(3) = \sigma_1^x \sigma_2^z \sigma_3^z \sigma_4^z \sigma_5^z \sigma_6^y - \sigma_1^y \sigma_2^z \sigma_3^z \sigma_4^z \sigma_5^z \sigma_6^x$$
$$-\sigma_2^x \sigma_3^z \sigma_4^z \sigma_5^y + \sigma_2^y \sigma_3^z \sigma_4^z \sigma_5^x + \sigma_3^x \sigma_4^y - \sigma_3^y \sigma_4^x. \tag{4.20}$$

The two families $\{H_n^{(1)}\}$ and $\{H_n^{(2)}\}$ do not commute with each other. In particular, the charges $\{H_n^{(2)}\}$ do not commute with $H_2^+$. Thus only half of the scalar and pseudoscalar families of the periodic chain can be modified by adding the boundary terms so that they still commute with the scalar Hamiltonian of the open chain. Both families are invariant under a global spin rotation around the $z$-axis: it is easily checked that all $H_n^{(i)}$'s commute with the generator of such rotation, the $z$-component of the total spin,

$$S^z = \sum_{j=1}^N \sigma_j^z. \tag{4.21}$$

Therefore, both families survive when the model is perturbed by a magnetic field term $hS^z$.

Consider now the anisotropic case (XY model):

$$H = \sum_{j=1}^{N-1} (\lambda_x \sigma_j^x \sigma_{j+1}^x + \lambda_y \sigma_j^y \sigma_{j+1}^y). \tag{4.22}$$

For the periodic XY chain, there are also two distinct families of conservation laws, all commuting together. One sequence is given by

$$^c H_n = \sum_{j=1}^N h_n^{XY}(j) \tag{4.23}$$

with

$$h_n^{XY}(j) = \lambda_x e_n^{xx}(j) + \lambda_y e_n^{yy}(j) + \lambda_x e_{n-2}^{yy}(j) + \lambda_y e_{n-2}^{xx}(j). \tag{4.24}$$

This expression gives the bulk density of the conserved charge $H_n$ for the open XY chain

$$H_n = H_n^{bulk} + h_n^L + h_n^R \tag{4.25}$$

$$H_n^{bulk} = \sum_{j=1}^{N-n+1} h_n^{XY}(j). \tag{4.26}$$

The border part is given by

$$h_n^L = - \sum_{k=1}^{[n/2]-1} \{ \lambda_x^{1-p_k} \lambda_y^{p_k} [e_{n-2k}^{xx}(k) + e_{n-2k}^{xx}(k-1)]$$
$$+ \lambda_x^{p_k} \lambda_y^{1-p_k} [e_{n-2k}^{yy}(k) + e_{n-2k}^{yy}(k-1)] \} \tag{4.27}$$

where

$$p_k = \tfrac{1}{2}(1 - (-1)^k) \tag{4.28}$$

with the convention

$$e_m^{\alpha\beta}(0) = 0. \tag{4.29}$$

For example,

$$h_8^{\mathrm{L}} = -\lambda_y e_6^{xx}(1) - \lambda_x e_6^{yy}(1) - \lambda_x e_4^{xx}(1) - \lambda_y e_4^{yy}(1) - \lambda_x e_4^{xx}(2)$$
$$-\lambda_y e_4^{yy}(2) - \lambda_y e_2^{xx}(3) - \lambda_y e_2^{xx}(2) - \lambda_x e_2^{yy}(3) - \lambda_x e_2^{yy}(2). \tag{4.30}$$

The commutativity of (4.25) with the Hamiltonian (4.22) can be established by a direct calculation. The second family of conservation laws (containing the pseudoscalar XX Hamiltonian) can be modified in a similar way to account for the anisotropic deformation.

In the presence of both the anisotropy and the magnetic field, i.e. for the open XYh model

$$H = \sum_{j=1}^{N-1} \lambda_x \sigma_j^x \sigma_{j+1}^x + \lambda_y \sigma_j^y \sigma_{j+1}^y + \sum_{j=1}^{N} h \sigma_j^z \tag{4.31}$$

except for the special cases $\lambda_x = \lambda_y$ (XXh model) and $\lambda_y = 0$ (the Xh model, equivalent to the Ising chain) there exist no counterterms that could be added to the bulk part of $H_3$ to account for the open boundary conditions. The first non-trivial charge in the non-degenerate case is $H_4$, with

$$H_4^{\mathrm{bulk}} = \sum_{j=1}^{N-3} (\lambda_x e_4^{xx}(j) + \lambda_y e_4^{yy}(j)) + \sum_{j=1}^{N-2} (\alpha_3 e_3^{xx}(j) + \beta_3 e_3^{yy}(j))$$
$$+ \sum_{j=1}^{N-1} (\alpha_2 e_2^{xx}(j) + \beta_2 e_2^{yy}(j)) + \kappa \sum_{j=1}^{N} \sigma_j^z \tag{4.32}$$

and

$$h_4^{\mathrm{L}} = -\lambda_y \sigma_1^x \sigma_2^x - \lambda_x \sigma_1^y \sigma_2^y - h(1 + \lambda_x/\lambda_y + \lambda_y/\lambda_x)\sigma_1^z \tag{4.33}$$

where

$$\alpha_3 = -h(\lambda_x + 2\lambda_y)/\lambda_y \tag{4.34}$$

$$\alpha_2 = -2\lambda_x - \lambda_x^2/\lambda_y + h^2/\lambda_x + h^2/\lambda_y + \lambda_x \kappa/h \tag{4.35}$$

and $\beta_i = \alpha_i(\lambda_x \leftrightarrow \lambda_y)$ ($i = 2, 3$), $\kappa$ being arbitrary.

We end this section with a remark on the open Hubbard model

$$H = -2 \sum_{s=\uparrow,\downarrow} \sum_{j=1}^{N-1} [(a_{j,s}^\dagger a_{j+1,s} + a_{j+1,s}^\dagger a_{j,s}) + 4U \sum_{j=1}^{N} (n_{j,\uparrow} - 1/2)(n_{j,\downarrow} - 1/2) \tag{4.36}$$

where $a_{j,s}^\dagger$ and $a_{j,s}$ are fermionic creation and annihilation operator of an electron of spin $s$ at site $j$, satisfying the anti-commutation relation

$$a_{j,s}^\dagger a_{k,s'} + a_{k,s'} a_{j,s}^\dagger = \delta_{j,k} \delta_{s,s'} \tag{4.37}$$

$U$ is a coupling constant and

$$n_{j,s} = a_{j,s}^\dagger a_{j,s}. \tag{4.38}$$

The integrability of this system has been established by means of the Bethe ansatz in [12]. As is well known, this model may be equivalently regarded as two coupled XX chains:

$$H = \sum_{j=1}^{N-1} [\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \tau_j^x \tau_{j+1}^x + \tau_j^y \tau_{j+1}^y] + U \sum_{j=1}^{N} \sigma_j^z \tau_j^z \tag{4.39}$$

(where $\sigma$ and $\tau$ denote two independent sets of Pauli matrices). The local conserved charges of the Hubbard model have been investigated in [1] for the case of periodic boundary conditions. For free open boundaries, the first non-trivial charge is found to be

$$H_4 = H_4^{\text{bulk}} + h_4^{\text{L}} + h_4^{\text{R}} \qquad (4.40)$$

with $H_4^{\text{bulk}}$ given by the restriction of the corresponding closed chain expression (see [1]), and

$$h_4^{\text{L}} = -(\sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y + \tau_1^x \tau_2^x + \tau_1^y \tau_2^y + 4U\sigma_1^z \tau_1^z). \qquad (4.41)$$

Unlike the XY case, no odd-order charges exist.

## 5. Further generalizations and conclusions

Our initial objective for this work was to obtain a closed form expression for the conserved charges of the open XXX chain, that is, to find the finite open chain deformation of the Catalan tree pattern of the conserved charges for the infinite chain. This has not been achieved, the main reason being that the spin polynomial basis used in the description of the charges in the closed case is inadequate for open chains. This, however, could be seen only after looking at the third non-trivial conservation law ($H_8$). The difficulties in studying the open chain conservation laws also come from the absence of a boost operator; there is thus no recursive way of constructing the charges.

In a sense, the problem has been solved half-way: we showed that the expressions for the charges split in two parts, a known bulk part and boundary corrections; we showed further how these corrections can be calculated systematically (and wrote down the leading boundary contribution).

Moreover, we have demonstrated that an open chain has half as many local conserved charges as its closed version with an identical number of sites. When one cuts open a closed spin chain, there appears an additional, symmetry (the chain reversal symmetry (2.22)), an immediate consequence of which being that all odd-order charges cease to be conserved. This is reminiscent of the reduction, by a factor 2, of the order of the two-dimensional conformal group induced by a boundary: in the presence of a boundary, the holomorphic and antiholomorphic Virasoro modes are no longer independent!

It should be stressed that the basic structure of the even-order conservation laws generated from the transfer matrix, i.e. the separation into the bulk and boundary parts, is a direct consequence of the construction (2.1)†. The structure (2.19) characterizes, therefore, a large class of integrable chains with non-trivial boundary conditions. Moreover, in models where the transfer matrix does not generate the complete family of conservation laws (in the XY model for instance), the additional conserved charges are expected to exhibit the same separation into bulk and boundary pieces.

The method for the calculation of the explicit expressions of the local conserved charges in the open XXX model presented in section 3 can be applied to more general situations in a straightforward way. For example, through a simple unitary transformation, the exact

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† As shown in section 2, all of the conserved charges generated by the transfer matrix of an open chain may be obtained in this way. Moreover, the chain reversal symmetry guarantees that our construction based on solving commutativity constraints for a boundary part added to the bulk part of an open chain charge, cannot succeed for odd orders. Therefore, our construction provides no more charges then the transfer matrix formalism. However, it is not immediately clear whether our approach (or the transfer matrix construction) exhausts all the possible conservation laws for open chain. The problem of completeness of the family of conserved charges generated by the transfer matrix is a very difficult one and has not yet been solved for generic (closed or open) integrable chains; it is certainly beyond the scope of our investigation.

expressions of $H_4$, $H_6$, and $H_8$ can be translated into conserved charges for the XXZ model with anisotropy parameter $\Delta = -1$ [2]. Similarly, by re-interpreting the vector product and the polynomials $f_{n,k}(j)$ in terms of a Lie algebra commutator, these expressions apply directly to the $su(N)$ version of the XXX chain, as well as to the octonionic chain [2]. All these chains are particular examples of a general algebraic model defined in [2], characterized by three arbitrary constants $\kappa_1$, $\kappa_2$ and $\kappa_3$. For periodic boundary conditions this model has been proved to be integrable provided that $\kappa_1 + \kappa_3 = 2\kappa_2$. The same condition assures also the integrability of this general model with free open boundary conditions. In particular, the boundary term in $H_4$ is

$$h_4^{\mathrm{L}} = (2\kappa_2/\kappa_1) f_{2,0}(1). \tag{5.1}$$

Although we have considered mainly open chains with free boundary conditions, we could treat more general boundary conditions in exactly the same way. Adding boundary terms in $H_2$ simply induces extra terms in the higher-order charges. For instance, by adding the boundary term $k\sigma_1^z$ to the Hamiltonian of the open XXX chain we find the following extra terms in $H_4$:

$$k\sigma_1^x\sigma_2^z\sigma_3^x + k\sigma_1^y\sigma_2^z\sigma_3^y - k\sigma_1^x\sigma_2^x\sigma_3^z - k\sigma_1^y\sigma_2^y\sigma_3^z - k^2\sigma_1^x\sigma_2^x - k^2\sigma_1^y\sigma_2^y - k\sigma_1^z + (3k - k^3)\sigma_1^z. \tag{5.2}$$

Anisotropic versions of the XXX model can be studied in similar way; again, the only difference is that the structure of conserved charges is more complicated. For instance, the first non-trivial conserved charge for the XYZ chain with free open boundaries is

$$H_4^{\mathrm{XYZ}} = H_4^{\mathrm{bulk}} + h_4^{\mathrm{L}} + h_4^{\mathrm{R}} \tag{5.3}$$

with $H_4^{\mathrm{bulk}}$ obtained from the corresponding closed XYZ expression (given in [1]) and

$$h_4^{\mathrm{L}} = \lambda_x(\lambda_y^2 + \lambda_z^2)\sigma_1^x\sigma_2^x + \lambda_y(\lambda_x^2 + \lambda_z^2)\sigma_1^y\sigma_2^y + \lambda_z(\lambda_x^2 + \lambda_y^2)\sigma_1^z\sigma_2^z. \tag{5.4}$$

For all these generalizations of the spin-$\frac{1}{2}$ XXX model, local conserved charges of odd degrees are also absent†. As already mentioned, this result is encoded in the transfer matrix formulation of the open chain models. Notice that when there are non-trivial boundary conditions, hence non-trivial matrices $\mathbf{X}_\pm$, the spin reversal transformation $\sigma_n \leftrightarrow \sigma_{N+1-n}$ must be accompanied by the interchange of $\mathbf{K}_+$ and $\mathbf{K}_-$. We expect that the same conclusion will hold true for higher-spin chains and even non-homogeneous chains with different spin representations at different sites (provide that the sequence of spaces $V_1, V_2, \ldots, V_N$ is invariant under the interchange of sites $n$ and $N + 1 - n$).

For the simpler XY model, we could obtain the exact expression for all the charges. Here again, there is no open chain deformation for half of the closed chain conserved charges, and as for closed chains, half of the charges cannot be obtained from the power expansion of the transfer matrix.

It is worth mentioning that for a particular choice of boundary terms the XXZ chain is endowed with quantum algebra symmetry $U_q[su(2)]$ [13]. Clearly, the corresponding family of local conserved charges is then invariant under the action of this quantum algebra. However, this additional symmetry does not seem to lead to a significant simplification in the pattern of the conserved quantities.

In [14], we have proposed a simple integrability test for closed chains with short-range interactions. This test essentially boils down to the existence of a conserved charge $H_3$ (more

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† One might ask whether there are some boundary conditions such that the only charges present, including the Hamiltonian, are of odd degree. However, this seems impossible: for example, for the XXX model with non-periodic boundary conditions, the lowest-order odd charges $H_3^{\mathrm{bulk}}$ and $H_5^{\mathrm{bulk}}$ cannot be deformed in a way that preserves their mutual commutativity.

precisely, we conjectured that the non-existence of $H_3$ means non-integrability). This test needs to be modified for open chains in an obvious way; here it is the existence of $H_4$ that should characterize the class of integrable models.

## Acknowledgments

## Appendix

For the sake of completeness, we give the basic commutators for the XXX model of the type $[\sigma_i \cdot \sigma_{i+1}, f_n(\mathcal{C})]$ (see equation (4.13) in [1]) in the graphical notation of section 3. In the expressions below, $\cdots$ stand for an arbitrary number of filled ($\bullet$) or open ($\circ$) dots.

$$\overset{\sqcap}{\underset{\circ}{}}\bullet \cdots \bullet = 2i \, \bullet \bullet \cdots \bullet \tag{6.1}$$

$$\overset{\sqcap}{\bullet}\circ \cdots \bullet = -2i \, \bullet \bullet \cdots \bullet \tag{6.2}$$

$$\overset{\sqcap}{\bullet}\bullet \cdots \bullet = 4i \, \circ \bullet \cdots \bullet - 4i \, \bullet \circ \cdots \bullet \tag{6.3}$$

$$\bullet \cdots \overset{\sqcap}{\bullet}\bullet \cdots \bullet = 2i \, \bullet \cdots \circ \bullet \cdots \bullet - 2i \, \bullet \cdots \bullet \circ \cdots \bullet \tag{6.4}$$

$$\bullet \cdots \overset{\sqcap}{\bullet}\circ \cdots \bullet = -2i \, \bullet \cdots ( \bullet \bullet ) \cdots \bullet. \tag{6.5}$$

In the last equation, the parentheses indicate a modification of the usual nesting pattern, e.g. the sequence $\bullet \bullet (\bullet \bullet) \bullet \bullet$, starting at site 1, say, corresponds to the polynomial $(((\sigma_1 \times \sigma_2) \times (\sigma_3 \times \sigma_4)) \times \sigma_5) \cdot \sigma_6$. Further basic commutators may be obtained applying the chain reversal symmetry to the rules (6.1)–(6.3) and (6.5). In particular, we have

$$\bullet \cdots \overset{\sqcap}{\bullet}\circ = -2i \, \bullet \cdots \bullet \bullet \tag{6.6}$$

$$\bullet \cdots \overset{\sqcap}{\circ}\bullet = 2i \, \bullet \cdots \bullet \bullet \tag{6.7}$$

$$\bullet \cdots \overset{\sqcap}{\bullet}\bullet = 4i \, \bullet \cdots \bullet \bullet - 4i \, \bullet \cdots \bullet \circ \tag{6.8}$$

$$\bullet \cdots \overset{\sqcap}{\circ}\bullet \cdots \bullet = 2i \, \bullet \cdots ( \bullet \bullet ) \cdots \bullet. \tag{6.9}$$

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